

Functions

Part Two

Outline for Today

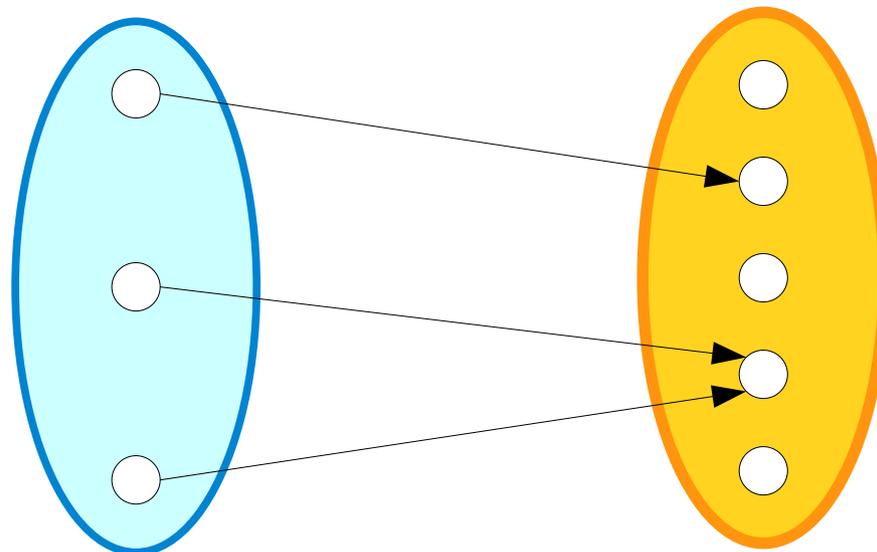
- ***Recap from Last Time***
 - Where are we, again?
- ***Injections and Surjections***
 - Two useful classes of functions.
- ***A Proof About Birds***
 - Trust me, it's relevant.
- ***Assuming vs Proving***
 - Two different roles to watch for.
- ***Connecting Function Types***
 - Relating the topics from last time.

Recap from Last Time

Domains and Codomains

- Every function f has two sets associated with it: its **domain** and its **codomain**.
- A function f can only be applied to elements of its domain. For any x in the domain, $f(x)$ belongs to the codomain.
- We write $f : A \rightarrow B$ to indicate that f is a function whose domain is A and whose codomain is B .

The function must be defined for each element of its domain.



The output of the function must always be in the codomain, but not all elements of the codomain need to be producible.

Involutions

- A function $f : A \rightarrow A$ from a set back to itself is called an ***involution*** if the following first-order logic statement is true about f :

$$\forall x \in A. f(f(x)) = x.$$

(“Applying f twice is equivalent to not applying f at all.”)

- For example, $f : \mathbb{R} \rightarrow \mathbb{R}$ defined as $f(x) = -x$ is an involution.

Injective Functions

- A function $f : A \rightarrow B$ is called **injective** (or **one-to-one**) if different inputs always map to different outputs.
 - A function with this property is called an **injection**.
- Formally, $f : A \rightarrow B$ is an injection if this FOL statement is true:

$$\forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \rightarrow f(a_1) \neq f(a_2))$$

(“If the inputs are different, the outputs are different”)

- Equivalently:

$$\forall a_1 \in A. \forall a_2 \in A. (f(a_1) = f(a_2) \rightarrow a_1 = a_2)$$

(“If the outputs are the same, the inputs are the same”)

	To <i>prove</i> that this is true...	
$\forall x. A$	Have the reader pick an arbitrary x . Then prove A is true for that choice of x .	
$\exists x. A$	Find an x where A is true. Then prove that A is true for that specific choice of x .	
$\neg A$	Simplify the negation, then consult this table on the result.	

New Stuff!

Proofs on Injections

Injective Functions

Theorem: Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be defined as $f(n) = 2n + 7$.
Then f is injective.

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Since $f(n_1) = f(n_2)$, we see that

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so $n_1 = n_2$, as required.

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Good exercise: Repeat this proof using the other definition of injectivity!

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$$\forall x_1 \in \mathbb{Z}. \forall x_2 \in \mathbb{Z}. (x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2))$$

What is the negation of this statement?

$$\neg \forall x_1 \in \mathbb{Z}. \forall x_2 \in \mathbb{Z}. (x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2))$$

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Therefore, we need to find $x_1, x_2 \in \mathbb{Z}$ such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$.

Can we do that?

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Let $x_1 = -1$ and $x_2 = +1$.

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$A \vee B$	Either prove $\neg A \rightarrow B$ or prove $\neg B \rightarrow A$. <i>(Why does this work?)</i>	
$A \leftrightarrow B$	Prove $A \rightarrow B$ and $B \rightarrow A$.	
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Another Class of Functions

Lassen Peak

Mt. Shasta

Crater Lake

Mt. McLoughlin

Mt. Hood

Mt. St. Helens

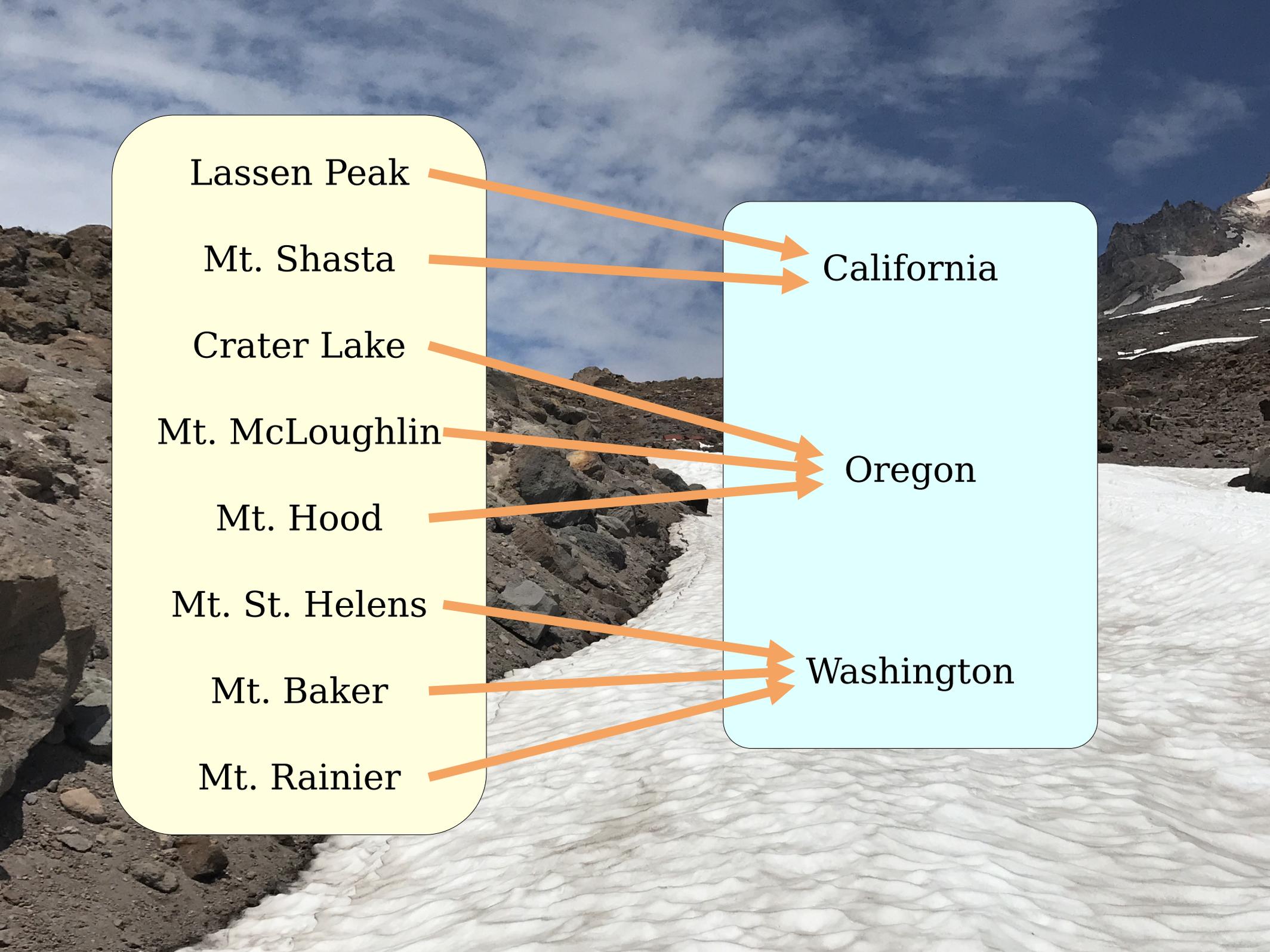
Mt. Baker

Mt. Rainier

California

Oregon

Washington



Surjective Functions

- A function $f : A \rightarrow B$ is called **surjective** (or **onto**) if this first-order logic statement is true about f :

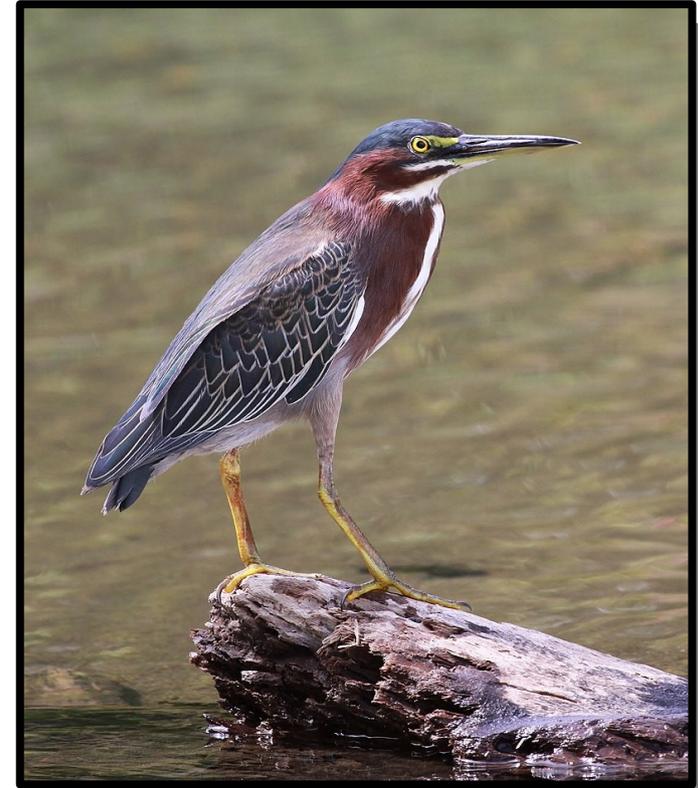
$$\forall b \in B. \exists a \in A. f(a) = b$$

(“For every output, there's an input that produces it.”)

- A function with this property is called a **surjection**.
- How does this compare to our first rule of functions?

Check the appendix for
sample proofs involving
injections.

A Proof About Birds



Theorem: If all birds can fly,
then all herons can fly.

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Given the predicates

Bird(*b*), which says *b* is a bird;

Heron(*h*), which says *h* is a heron; and

CanFly(*x*), which says *x* can fly,

translate the theorem into first-order logic.

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translate the theorem into first-order logic.

$(\forall b. (Bird(b) \rightarrow CanFly(b))) \rightarrow (\forall h. (Heron(h) \rightarrow CanFly(h)))$

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All herons
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Proof: Assume that all birds can fly. We will show that all herons can fly.

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Which makes more sense as the next step in this proof?

1. Consider an arbitrary bird b .
2. Consider an arbitrary heron h .

$(\forall b. (Bird(b) \rightarrow CanFly(b))) \rightarrow (\forall h. (Heron(h) \rightarrow CanFly(h)))$



All birds can fly

All herons can fly

Theorem: If all birds can fly, then all herons can fly.

Proof: Assume that all birds can fly. We will show that all herons can fly.

Which makes more sense as the next step in this proof?

1. Consider an arbitrary bird b .
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Consider an arbitrary bird b . Since b is a bird, b can fly.

$$\underbrace{(\forall b. (Bird(b) \rightarrow CanFly(b)))}_{\text{All birds can fly}} \rightarrow \underbrace{(\forall h. (Heron(h) \rightarrow CanFly(h)))}_{\text{All herons can fly}}$$

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Proof: Assume that all birds can fly. We will show that all herons can fly.

Consider an arbitrary bird b . Since b is a bird, b can fly. *[and now we're stuck! we are interested in herons, but b might not be one. It could be a hummingbird, for example!]*

$$\underbrace{(\forall b. (Bird(b) \rightarrow CanFly(b)))}_{\text{All birds can fly}} \rightarrow \underbrace{(\forall h. (Heron(h) \rightarrow CanFly(h)))}_{\text{All herons can fly}}$$

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Proof: Assume that all birds can fly. We will show that all herons can fly.

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$$\underbrace{(\forall b. (Bird(b) \rightarrow CanFly(b)))}_{\text{All birds can fly}} \rightarrow \underbrace{(\forall h. (Heron(h) \rightarrow CanFly(h)))}_{\text{All herons can fly}}$$

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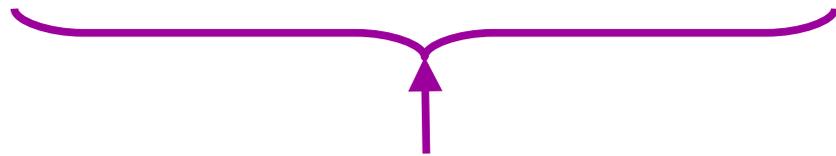
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$(\forall b. (Bird(b) \rightarrow CanFly(b))) \rightarrow (\forall h. (Heron(h) \rightarrow CanFly(h)))$



We never introduce a variable b .



We introduce a variable h almost immediately.

Proving vs. Assuming

- In the context of a proof, you will need to assume some statements and prove others.
 - Here, we **assumed** all birds can fly.
 - Here, we **proved** all herons can fly.
- Statements behave differently based on whether you're assuming or proving them.

$(\forall b. (Bird(b) \rightarrow CanFly(b))) \rightarrow (\forall h. (Heron(h) \rightarrow CanFly(h)))$



We never introduce a variable b .



We introduce a variable h almost immediately.

Proving vs. Assuming

- To **prove** the universally-quantified statement

$$\forall x. P(x)$$

we introduce a new variable x representing some arbitrarily-chosen value.

- Then, we prove that $P(x)$ is true for that variable x .
- That's why we introduced a variable h in this proof representing a heron.

$$(\forall b. (Bird(b) \rightarrow CanFly(b))) \rightarrow (\forall h. (Heron(h) \rightarrow CanFly(h)))$$

We never introduce a variable b .

We introduce a variable h almost immediately.

Proving vs. Assuming

- If we **assume** the statement

$$\forall x. P(x)$$

we **do not** introduce a variable x .

- Rather, if we find a relevant value z somewhere else in the proof, we can conclude that $P(z)$ is true.
- That's why we didn't introduce a variable b in our proof, and why we concluded that h , our heron, can fly.

$$(\forall b. (Bird(b) \rightarrow CanFly(b))) \rightarrow (\forall h. (Heron(h) \rightarrow CanFly(h)))$$

We never introduce a variable b .

We introduce a variable h almost immediately.

	To <i>prove</i> that this is true...	
$\forall x. A$	Have the reader pick an arbitrary x . We then prove A is true for that choice of x .	
$\exists x. A$	Find an x where A is true. Then prove that A is true for that specific choice of x .	
$A \rightarrow B$	Assume A is true, then prove B is true.	
$A \wedge B$	Prove A . Then prove B .	
$A \vee B$	Either prove $\neg A \rightarrow B$ or prove $\neg B \rightarrow A$. <i>(Why does this work?)</i>	
$A \leftrightarrow B$	Prove $A \rightarrow B$ and $B \rightarrow A$.	
$\neg A$	Simplify the negation, then consult this table on the result.	

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$\forall x. A$	Have the reader pick an arbitrary x . We then prove A is true for that choice of x .	Initially, do nothing . Once you find a z through other means, you can state it has property A .
$\exists x. A$	Find an x where A is true. Then prove that A is true for that specific choice of x .	
$A \rightarrow B$	Assume A is true, then prove B is true.	Initially, do nothing . Once you know A is true, you can conclude B is also true.
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$\exists x. A$	Find an x where A is true. Then prove that A is true for that specific choice of x .	Introduce a variable x into your proof that has property A .
$A \rightarrow B$	Assume A is true, then prove B is true.	Initially, do nothing . Once you know A is true, you can conclude B is also true.
$A \wedge B$	Prove A . Then prove B .	Assume A . Then assume B .
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$\exists x. A$	Find an x where A is true. Then prove that A is true for that specific choice of x .	Introduce a variable x into your proof that has property A .
$A \rightarrow B$	Assume A is true, then prove B is true.	Initially, do nothing . Once you know A is true, you can conclude B is also true.
$A \wedge B$	Prove A . Then prove B .	Assume A . Then assume B .
$A \vee B$	Either prove $\neg A \rightarrow B$ or prove $\neg B \rightarrow A$. <i>(Why does this work?)</i>	Consider two cases. Case 1: A is true. Case 2: B is true.
$A \leftrightarrow B$	Prove $A \rightarrow B$ and $B \rightarrow A$.	Assume $A \rightarrow B$ and $B \rightarrow A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

Connecting Function Types

Types of Functions

- We now have three special types of functions:
 - ***involutions***, functions that undo themselves;
 - ***injections***, functions where different inputs go to different outputs; and
 - ***surjections***, functions that cover their whole codomain.
- ***Question:*** How do these three classes of functions relate to one another?

Theorem: For any function $f : A \rightarrow A$, if f is an involution, then f is surjective.

$$\underbrace{(\forall x \in A. f(f(x)) = x)}_{\substack{f \text{ is an} \\ \text{involution.}}} \rightarrow \underbrace{(\forall b \in A. \exists a \in A. f(a) = b)}_{\substack{f \text{ is} \\ \text{surjective.}}}$$

Theorem: For any function $f : A \rightarrow A$, if f is an involution, then f is surjective.

$$(\forall x \in A. f(f(x)) = x) \rightarrow (\forall b \in A. \exists a \in A. f(a) = b)$$

Assume this.

Prove this.

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If you ***assume***
this is true...

Initially, ***do nothing***. Once you
find a z through other means,
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if f is an involution, then f is surjective.

$$(\forall x \in A. f(f(x)) = x) \rightarrow (\forall b \in A. \exists a \in A. f(a) = b)$$

Assume this.

Prove this.

Since we're assuming this, we aren't going to pick a specific choice of x right now. Instead, we're going to keep an eye out for something to apply this fact to.

Proof Outline

1. Assume f is an involution.

Theorem: For any function $f : A \rightarrow A$, if f is an involution, then f is surjective.

$$(\forall x \in A. f(f(x)) = x) \rightarrow (\forall b \in A. \exists a \in A. f(a) = b)$$

Ass

We've said that we need to prove this statement. How do we do that?

Prove this.

Proof Outline

1. Assume f is an involution.

Theorem: For any function $f : A \rightarrow A$, if f is an involution, then f is surjective.

$$(\forall x \in A. f(f(x)) = x) \rightarrow (\forall b \in A. \exists a \in A. f(a) = b)$$

Assume this

To **prove** that
this is true...

Have the reader pick an
arbitrary x . Then prove A is
true for that choice of x .

Prove this.

Proof Outline

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There's a universal quantifier up front. Since we're proving this, we'll pick an arbitrary $b \in A$.

Prove this.

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$$(\forall x \in A. f(f(x)) = x) \rightarrow (\forall b \in A. \exists a \in A. f(a) = b)$$

Now, we hit an existential quantifier. Since we're proving this, we need to find a choice of $a \in A$ where this is true.

Prove this.

Proof Outline

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2. Pick an arbitrary $b \in A$.

Theorem: For any function $f : A \rightarrow A$, if f is an involution, then f is surjective.

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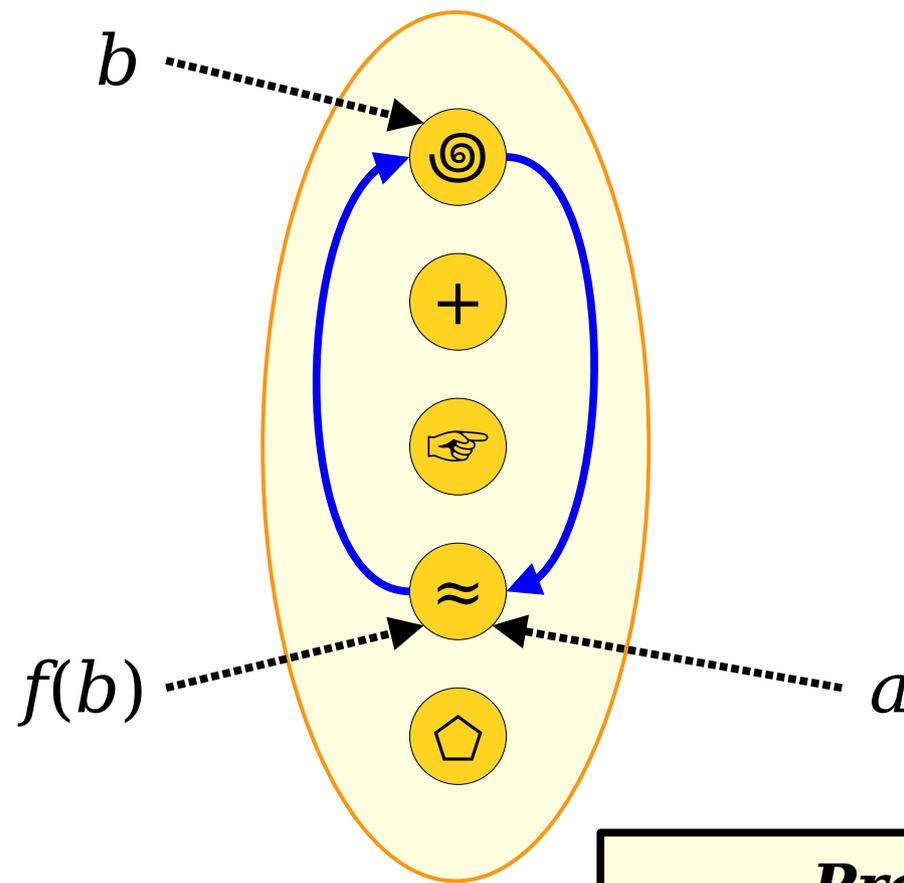
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Prove this.

Proof Outline

1. Assume f is an involution.
2. Pick an arbitrary $b \in A$.
3. Give a choice of $a \in A$ where $f(a) = b$.

Theorem: For any function $f : A \rightarrow A$, if f is an involution, then f is surjective.



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Proof: Pick any involution $f : A \rightarrow A$.

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Theorem: For any function $f : A \rightarrow A$, if f is an involution, then f is surjective.

Proof: Pick any involution $f : A \rightarrow A$. We will prove that f is surjective.

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Specifically, pick $a = f(b)$.

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This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

Theorem: For any function $f : A \rightarrow A$, if f is an involution, then f is injective.

Proof: In the slide appendix!

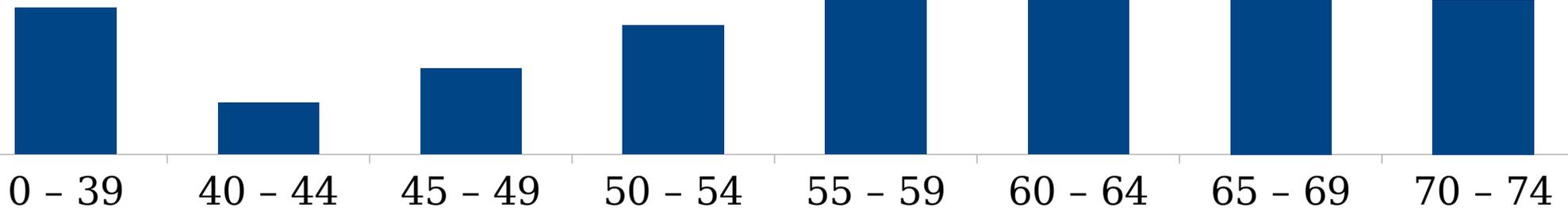
Time-Out for Announcements!

Problem Set One Graded

- Your wonderful TAs have finished grading Problem Set One.
- Grades and feedback are up on the Gradescope.
- Solutions are available online on the course website (visit the page for PS1 to get the link).

Problem Set One Graded

75th Percentile: **66 / 74 (89%)**
50th Percentile: **61 / 74 (82%)**
25th Percentile: **55 / 74 (74%)**



Pro tips when reading a grading distribution:

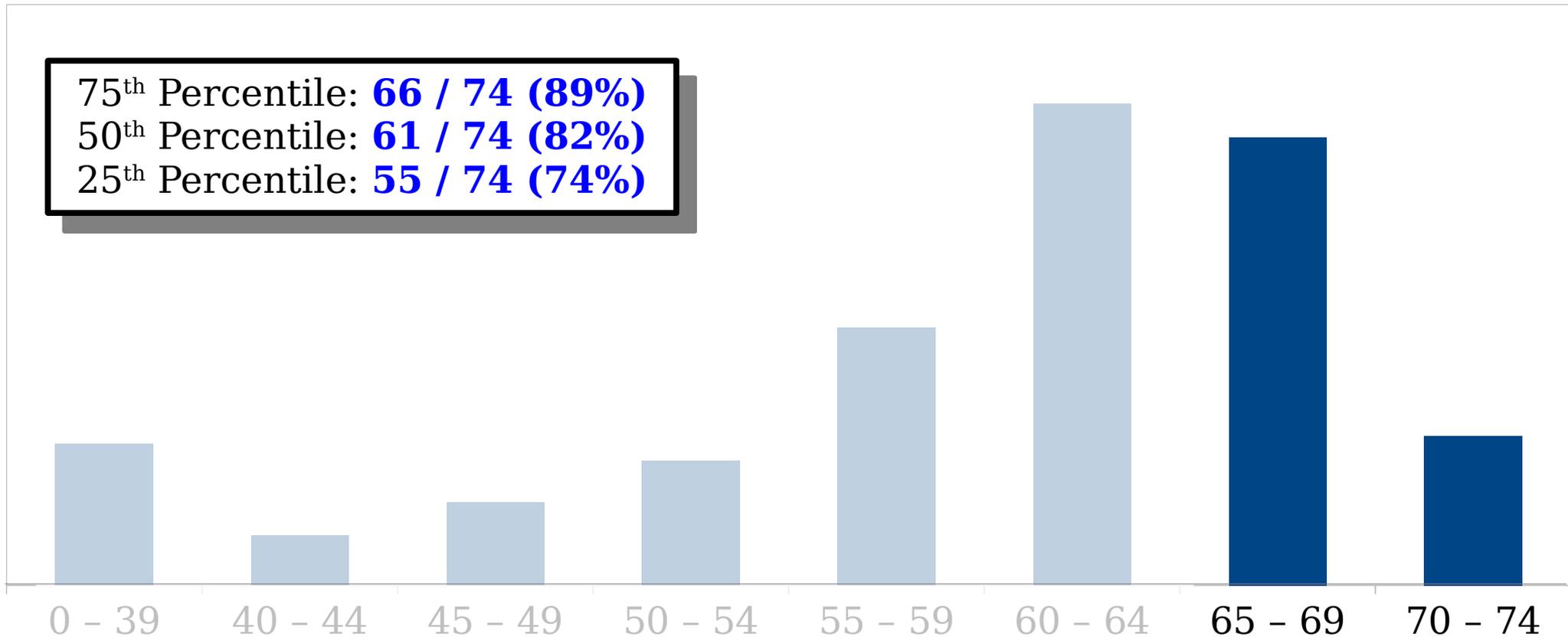
1. Standard deviations are *unhelpful and discouraging*. Ignore them.
2. The average score is a *unhelpful*. Ignore it.
3. Raw scores are *unhelpful and discouraging*. Ignore them.

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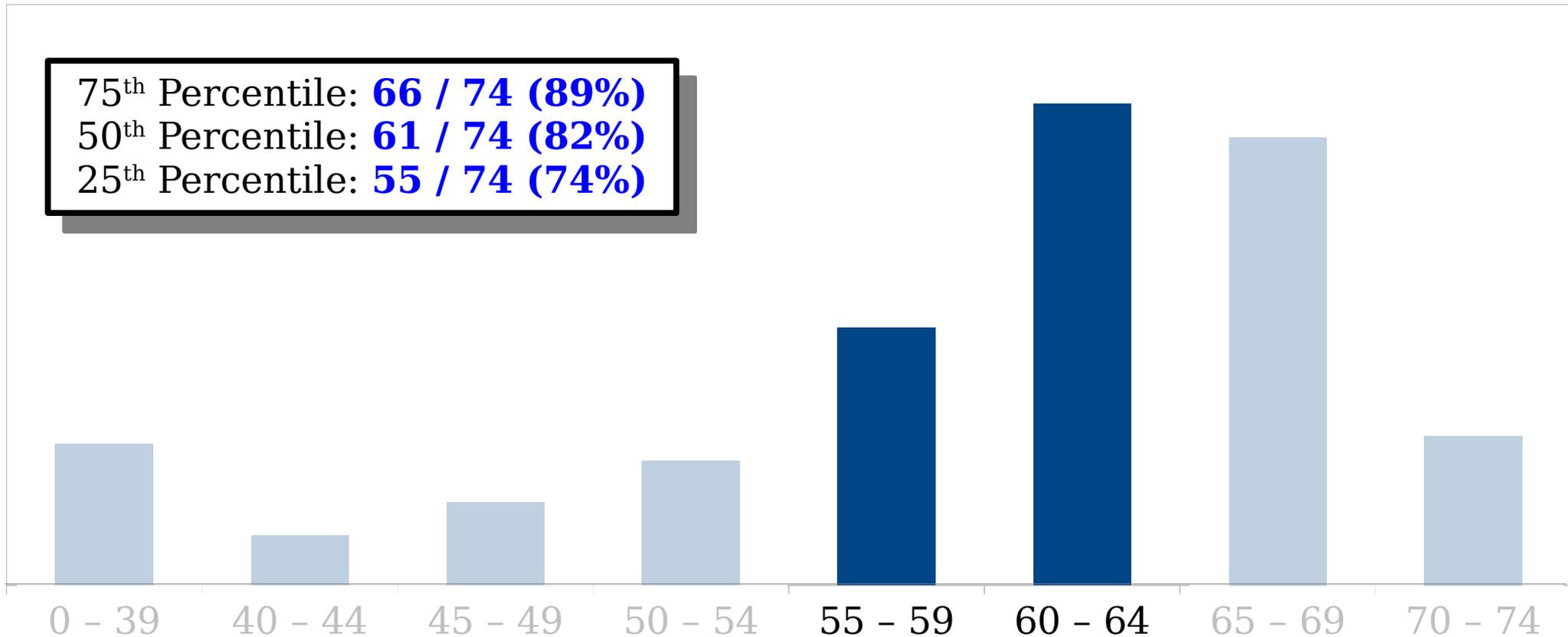
"Great job! Look over your feedback for some tips on how to tweak things for next time."

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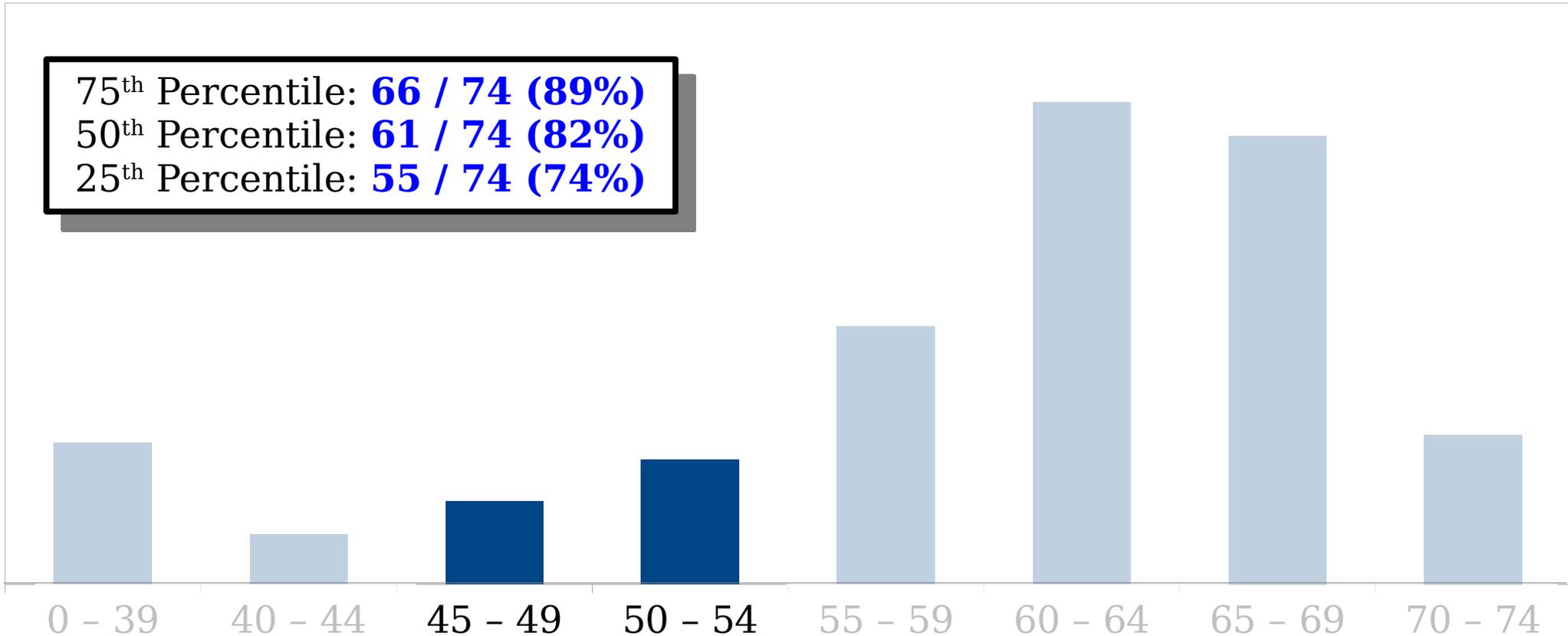
"You're almost there! Review the feedback on your submission and see what to focus on for next time."

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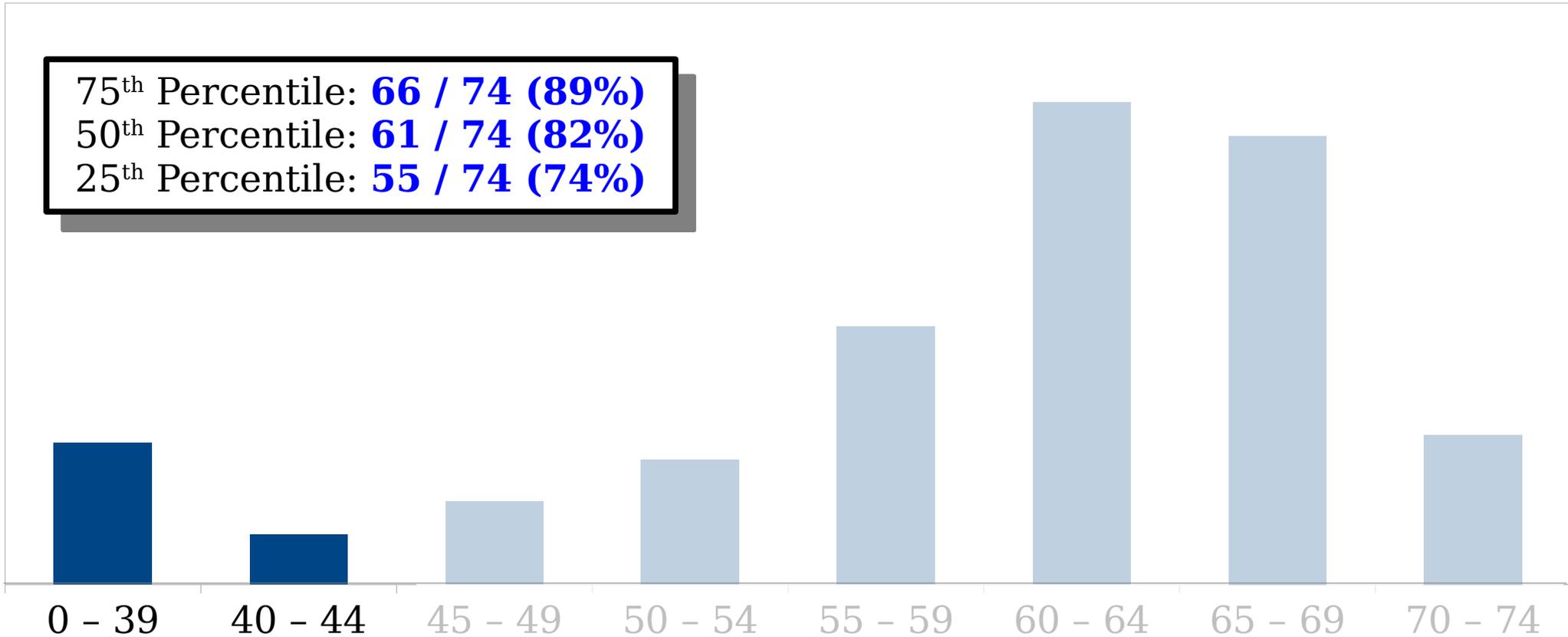
25th Percentile: **55 / 74 (74%)**



"You're on the right track, but there are some areas where you need to improve. Review your feedback and ask us questions when you have them."

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"Looks like something hasn't quite clicked yet. Get in touch with us and stop by office hours to get some extra feedback and advice. Don't get discouraged - you can do this!"

What Not to Think

- “Well, I guess I’m just not good at math.”
 - For most of you, this is your first time doing any rigorous proof-based math.
 - Don’t judge your future performance based on a single data point.
 - Life advice: think about download times.
 - Life advice: have a growth mindset!
- “Hey, I did above the median. That’s good enough.”
 - There’s always some area where you can improve. Take the time to see what that is.

Regrade Requests

- We're human. We make mistakes. And we're happy to correct them!
- Regrades will open on Gradescope 48 hours after grades are released. They close one week after grades are released.
- Notes on regrades:
 - Please be civil. We make mistakes. We're happy to correct them.
 - We have to grade what you submitted; we can't take any clarifications into account during regrades.
 - Regrades are for where we made deductions we shouldn't have, rather than for the magnitude of deductions.

Your Questions

“How are concepts we learn in CS103 applicable to fields outside of CS and math, such as physics, chemistry, or materials science?”

Functions that are both injective and surjective (bijections – a term we’ll see Friday) are useful tools in pinning down what symmetry means. That has applications throughout a bunch of other fields, especially physics and Matsci.

Graphs, which we’ll cover next week, are indispensable tools in a bunch of different disciplines. They’re used to model discrete lattice structures in physics, molecules in chemistry, etc.

Even the limits of computation have applications in the sciences. It’s not computationally possible to determine how arbitrary crystals assemble, nor can we determine some physical properties of quantum systems with computers.

Complexity theory is particularly relevant in many disciplines, as some problems that seem easy to state are computationally hard to solve. This dictates what sorts of questions folks then ask.

Major Ideas From Today

- Proofs involving first-order definitions are heavily based on the structure of those definitions, yet FOL notation itself does *not* appear in the proof.
- Statements behave differently based on whether you're **assuming** or **proving** them.
- When you **assume** a universally-quantified statement, initially, do nothing. Instead, keep an eye out for a place to apply the statement more specifically.
- When you **prove** a universally-quantified statement, pick an arbitrary value and try to prove it has the needed property.

	To <i>prove</i> that this is true...	If you <i>assume</i> this is true...
$\forall x. A$	Have the reader pick an arbitrary x . We then prove A is true for that choice of x .	Initially, do nothing . Once you find a z through other means, you can state it has property A .
$\exists x. A$	Find an x where A is true. Then prove that A is true for that specific choice of x .	Introduce a variable x into your proof that has property A .
$A \rightarrow B$	Assume A is true, then prove B is true.	Initially, do nothing . Once you know A is true, you can conclude B is also true.
$A \wedge B$	Prove A . Then prove B .	Assume A . Then assume B .
$A \vee B$	Either prove $\neg A \rightarrow B$ or prove $\neg B \rightarrow A$. <i>(Why does this work?)</i>	Consider two cases. Case 1: A is true. Case 2: B is true.
$A \leftrightarrow B$	Prove $A \rightarrow B$ and $B \rightarrow A$.	Assume $A \rightarrow B$ and $B \rightarrow A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

Next Time

- ***Cardinality Revisited***
 - Formalizing our definitions.
- ***The Nature of Infinity***
 - Infinity is more interesting than it looks!
- ***Cantor's Theorem Revisited***
 - Formally proving a major result.

Appendix: More Proofs on Functions

Proof 1: Proving a function is surjective.

Surjective Functions

Theorem: Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

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Proof:

What does it mean for f to be surjective?

$$\forall y \in \mathbb{R}. \exists x \in \mathbb{R}. f(x) = y$$

Therefore, we'll choose an arbitrary $y \in \mathbb{R}$, then prove that there is some $x \in \mathbb{R}$ where $f(x) = y$.

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Let $x = y / 2$.

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So $f(x) = y$, as required. ■

This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

Proof 2: Proving a function is not surjective.

Surjective Functions

Theorem: Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

Surjective Functions

Theorem: Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

What does it mean for g to be surjective?

$$\forall n \in \mathbb{N}. \exists m \in \mathbb{N}. g(m) = n$$

What is the negation of the above statement?

$$\neg \forall n \in \mathbb{N}. \exists m \in \mathbb{N}. g(m) = n$$

$$\exists n \in \mathbb{N}. \neg \exists m \in \mathbb{N}. g(m) = n$$

$$\exists n \in \mathbb{N}. \forall m \in \mathbb{N}. g(m) \neq n$$

Therefore, we need to find a natural number n where, regardless of which m we pick, we have $g(m) \neq n$.

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Proof: Let $n = 137$.

Our overall goal is to prove

$$\exists n \in \mathbb{N}. \forall m \in \mathbb{N}. g(m) \neq n.$$

We just made our choice of n .

Therefore, we need to prove

$$\forall m \in \mathbb{N}. g(m) \neq n.$$

We'll therefore pick an arbitrary $m \in \mathbb{N}$, then prove that $g(m) \neq n$.

Surjective Functions

Theorem: Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

Proof: Let $n = 137$. Now, pick an arbitrary $m \in \mathbb{N}$.

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Notice that $g(m) = 2m$ is even, while 137 is odd.

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Proof 3: Proving all involutions
are injective.

$$(\forall x \in A. f(f(x)) = x) \rightarrow (\forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \rightarrow f(a_1) \neq f(a_2)))$$

f is an
involution.

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Theorem: For any function $f : A \rightarrow A$,
if f is an involution, then f is injective.

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$$(\forall b. (Bird(b) \rightarrow CanFly(b))) \rightarrow (\forall h. (Heron(h) \rightarrow CanFly(h)))$$

Assume this.

Prove this.

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Assume
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Prove
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Since we're assuming this, we aren't going to pick a specific choice of x right now. Instead, we're going to keep an eye out for something to apply this fact to.

Proof Outline

1. Assume f is an involution.

Theorem: For any function $f : A \rightarrow A$, if f is an involution, then f is injective.

$$(\forall x \in A. f(f(x)) = x) \rightarrow (\forall a_1 \in A. \forall a_2 \in A. (a_1 \neq a_2 \rightarrow f(a_1) \neq f(a_2)))$$

We need to prove this part.
What does that mean?

Prove
this.

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Since we're proving something universally-quantified, we'll pick some values arbitrarily.

Prove this.

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We now need to prove this implication. But we know how to do that! We assume the antecedent and prove the consequent.

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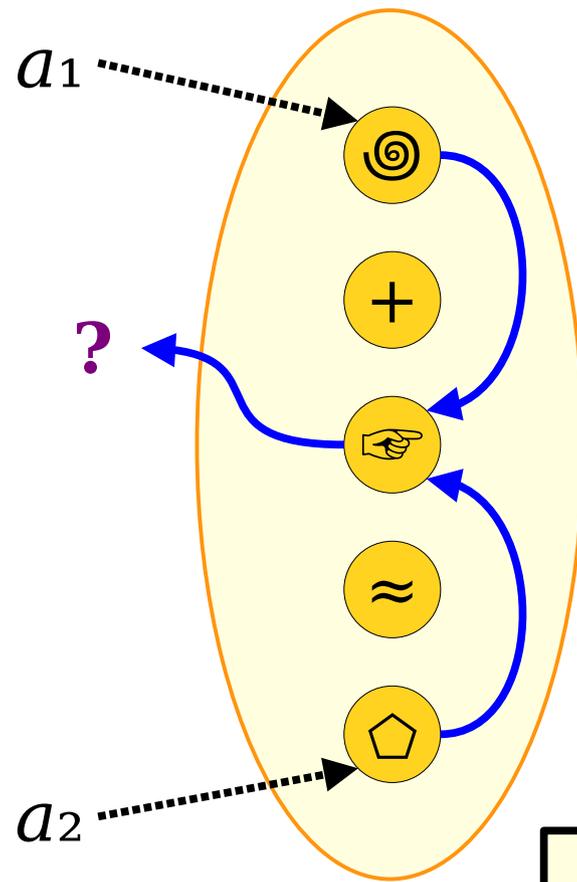
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